PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On a Special Form of the General Equation of a Cubic Surface and on a Diagram Representing the Twenty-Seven Lines on the Surface

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II. *On a Special Form of the General Equation of a Cubic Surface and on a Diagram Representing the Twenty-seven Lines on the Surface.**

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THE existence of straight lines on a cubic surface, the number of them, and their relations to each other was first discussed in a correspondence between SALMON and CAYLEY.

In a paper which appeared in 1849, in vol. 4 of the 'Cambridge and Dublin Mathematical Journal,' "On the Triple Tangent Planes of Surfaces of the Third Order," CAYLEY gave a sketch of what was then known, and gave the equations of the forty-five planes in which the twenty-seven lines on the surface lie by threes, when the equation of the surface is taken in a particular form.

In the above-mentioned paper, CAYLEY remarks, "there is great difficulty in conceiving the complete figure formed by the twenty-seven lines: indeed, this can hardly, I think, be accomplished until a more perfect notation is discovered."

SCHLÄFLI† has discovered a notation of great merit which affords a powerful method of dealing with the twenty-seven lines; it is based upon the selection of some twelve of the lines which form a "double six." The author of this paper endeavoured to find a notation for the twenty-seven lines, which did not depend on any special selection among them. He hopes that the method he has adopted of representing by a plane diagram the intersection or non-intersection of the twenty-seven lines with each other will be found of some interest.

Four distinct forms of the diagram are given: one will be found of more use for one purpose, and another for another; although each contains everything that is contained in the others. In fact, one is obtained from another by purely clerical alteration.

The contents of this paper may be stated shortly as follows:—

In § 1 it is shown that the equation of the general cubic surface may be thrown into the form

* As originally communicated, this paper was entitled, "On a Graphical Representation of the Twenty-seven Lines on a Cubic Surface."

† 'Quarterly Journal of Mathematics,' vol. 2, p. 116.

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$$\text{KLMN} = (\text{T} - \text{K})(\text{T} - \text{L})(\text{T} - \text{M})(\text{T} - \text{N}),$$

where K, L, M, N, T equated to zero represent planes.

In §§ 2–9, it is shown how to obtain the equations of the twenty-seven lines on the surface whose equation is

$$xyzu = (x - a\text{T})(y - b\text{T})(z - c\text{T})(u - d\text{T}),$$

and further it is shown which of the twenty-seven lines intersect each other.

In § 10 the method of representation by a plane-diagram is explained, and the remaining part of the paper consists chiefly in deducing mutual relations between the lines by means of the diagram or one of its transformations.

It may be explained that of the four transformations of the diagram, Figure A is arranged to show that the lines which are numbered 1 to 15 form in threes, five triangles; the remaining 12 lines, which are numbered 16 to 27, do not form a single triangle by themselves.*

Figure B is arranged to show that not only can nine planes be drawn to pass through all the twenty-seven lines, but that they can be arranged in three sets of nine each, such that each set forms three triangles in two distinct ways.

Figure C is arranged to exhibit what is called a "double six" in the left hand top corner. It is of use for observing what lines intersect or do not intersect a number of non-intersecting straight lines, such as the six numbered 20, 21, 8, 11, 3, 4, or the six numbered 26, 27, 5, 2, 9, 10.

Figure D is arranged to show that it is possible to form a closed polygon of all the twenty-seven lines, such that no side intersects either of the sides next but one to itself.

This figure is of use for observing what lines intersect, or do not intersect, the sides of a closed quadrilateral, pentagon, or hexagon, such as are formed by the lines numbered 26, 17, 1, 19; 16, 23, 26, 17, 1, and 2, 3, 10, 11, 9, 4 respectively.

* It has been remarked as an omission in this paper that the fact that these twelve lines form a "double six" is nowhere stated.

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FIGURE A.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	
27	·	·	*	*	·	·	·	*	·	·	*	·	·	·	*	*	·	*	·	*	·	*	·	*	·	·	o	27
26	·	·	*	*	·	·	·	*	·	·	*	·	·	·	*	·	*	·	*	·	*	·	*	·	*	o	·	26
25	·	·	*	·	·	*	·	·	*	·	·	*	·	*	·	*	·	*	·	*	·	*	·	·	o	*	·	25
24	·	·	*	·	·	*	·	·	*	·	·	*	·	*	·	·	*	·	*	·	*	·	*	o	·	·	*	24
23	·	*	·	·	·	*	*	·	·	·	*	·	*	·	·	*	·	*	·	*	·	·	o	*	·	*	·	23
22	·	*	·	·	·	*	*	·	·	·	*	·	*	·	·	·	*	·	*	·	*	o	·	·	*	·	*	22
21	·	*	·	·	*	·	·	·	*	*	·	·	·	·	*	*	·	*	·	·	o	*	·	*	·	*	·	21
20	·	*	·	·	*	·	·	·	*	*	·	·	·	·	*	·	*	·	*	o	·	·	*	·	*	·	*	20
19	*	·	·	·	*	·	·	*	·	·	·	*	*	·	·	*	·	·	o	*	·	*	·	*	·	*	·	19
18	*	·	·	·	*	·	·	*	·	·	·	*	*	·	·	·	*	o	·	·	*	·	*	·	*	·	*	18
17	*	·	·	*	·	·	*	·	·	*	·	·	·	*	·	·	o	*	·	*	·	*	·	*	·	*	·	17
16	*	·	·	*	·	·	*	·	·	*	·	·	·	*	·	o	·	·	*	·	*	·	*	·	*	·	*	16
15	*	·	·	·	·	*	*	·	·	·	·	*	*	*	o	·	·	·	·	*	*	·	·	·	·	*	*	15
14	·	*	·	·	*	·	·	*	·	·	*	·	*	o	*	*	*	·	·	·	·	·	·	*	*	·	·	14
13	·	·	*	*	·	·	·	·	*	*	·	·	o	*	*	·	·	*	*	·	·	*	*	·	·	·	·	13
12	·	*	·	*	·	·	*	·	·	*	*	o	·	·	*	·	·	*	*	·	·	·	·	*	*	·	·	12
11	*	·	·	·	*	·	·	·	*	*	o	*	·	*	·	·	·	·	·	·	·	*	*	·	·	*	*	11
10	·	·	*	·	·	*	·	*	·	o	*	*	*	·	·	*	*	·	·	*	*	·	·	·	·	·	·	10
9	*	·	·	*	·	·	*	*	o	·	*	·	*	·	·	·	·	·	·	*	*	·	·	*	*	·	·	9
8	·	*	·	·	·	*	*	o	*	*	·	·	·	*	·	·	·	*	*	·	·	·	·	·	·	*	*	8
7	·	·	*	·	*	·	o	*	*	·	·	*	·	·	*	*	*	·	·	·	·	*	*	·	·	·	·	7
6	*	·	·	*	*	o	·	*	·	*	·	·	·	·	*	·	·	·	·	·	·	*	*	*	*	·	·	6
5	·	·	*	*	o	*	*	·	·	·	*	·	·	*	·	·	·	*	*	*	*	·	·	·	·	·	·	5
4	·	*	·	o	*	*	·	·	*	·	·	*	*	·	·	*	*	·	·	·	·	·	·	·	·	*	*	4
3	*	*	o	·	*	·	*	·	·	*	·	·	*	·	·	·	·	·	·	·	·	·	·	*	*	*	*	3
2	*	o	*	*	·	·	·	*	·	·	·	*	·	*	·	·	·	·	·	*	*	*	*	·	·	·	·	2
1	o	*	*	·	·	*	·	·	*	·	*	·	·	·	*	*	*	*	*	·	·	·	·	·	·	·	·	1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	

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FIGURE B.

	4	6	5	9	8	7	13	10	3	26	21	15	11	22	27	23	2	20	16	25	14	1	18	17	19	12	24	
24	·	*	·	*	·	·	·	·	*	·	*	·	·	·	*	*	·	·	·	·	*	·	·	*	*	*	0	24
12	*	·	·	·	·	*	·	*	·	·	·	*	*	·	·	·	*	·	·	*	·	·	*	·	*	0	*	12
19	·	·	*	·	*	·	*	·	·	*	·	·	·	*	·	·	·	*	*	·	·	*	·	·	0	*	*	19
17	*	·	·	·	·	*	·	*	·	*	·	·	·	*	·	·	·	*	·	·	*	*	*	0	·	·	*	17
18	·	·	*	·	*	·	*	·	·	·	*	·	·	·	*	*	·	·	·	*	·	*	0	*	·	*	·	18
1	·	*	·	*	·	·	·	·	*	·	·	*	*	·	·	·	*	·	*	·	·	0	*	*	*	·	·	1
14	·	·	*	·	*	·	*	·	·	·	·	*	*	·	·	·	*	·	*	*	0	·	·	*	·	·	*	14
25	·	*	·	*	·	·	·	·	*	*	·	·	·	*	·	·	·	*	*	0	*	·	*	·	·	*	·	25
16	*	·	·	·	·	*	·	*	·	·	*	·	·	·	*	*	·	·	0	*	*	*	·	·	*	·	·	16
20	·	·	*	*	·	·	·	*	·	·	·	*	·	·	*	*	*	0	·	*	·	·	·	*	*	·	·	20
2	*	·	·	·	*	·	·	·	*	·	*	·	·	*	·	*	0	*	·	·	*	*	·	·	·	*	·	2
23	·	*	·	·	·	*	*	·	·	*	·	·	*	·	·	0	*	*	*	·	·	·	*	·	·	·	*	23
27	*	·	·	·	*	·	·	·	*	·	·	*	*	*	0	·	·	*	*	·	·	·	*	·	·	·	*	27
22	·	*	·	·	·	*	*	·	·	·	*	·	*	0	*	·	*	·	·	*	·	·	·	*	*	·	·	22
11	·	·	*	*	·	·	·	*	·	*	·	·	0	*	*	*	·	·	·	·	*	*	·	·	·	*	·	11
15	·	*	·	·	·	*	*	·	·	*	*	0	·	·	*	·	·	*	·	·	*	*	·	·	·	*	·	15
21	·	·	*	*	·	·	·	*	·	*	0	*	·	*	·	·	*	·	*	·	·	·	*	·	·	·	*	21
26	*	·	·	·	*	·	·	·	*	0	*	*	*	·	·	*	·	·	·	*	·	·	·	*	*	·	·	26
3	·	·	*	·	·	*	*	*	0	*	·	·	·	·	*	·	*	·	·	*	·	*	·	·	·	·	*	3
10	·	*	·	·	*	·	*	0	*	·	*	·	*	·	·	·	·	*	*	·	·	·	·	*	·	*	·	10
13	*	·	·	*	·	·	0	*	*	·	·	*	·	*	·	*	·	·	·	·	*	·	*	·	*	·	·	13
7	·	·	*	*	*	0	·	·	*	·	·	*	·	*	·	*	·	·	*	·	·	·	·	*	·	*	·	7
8	·	*	·	*	0	*	·	*	·	*	·	·	·	·	*	·	*	·	·	·	*	·	*	·	*	·	·	8
9	*	·	·	0	*	*	*	·	·	·	*	·	*	·	·	·	·	*	·	*	·	*	·	·	·	·	*	9
5	*	*	0	·	·	*	·	·	*	·	*	·	*	·	·			*	·	·	*	·	*	·	*	·	·	5
6	*	0	*	·	*	·	·	*	·	·	·	*	·	*	·	*			·	*	·	*	·	·	·	·	*	6
4	0	*	*	*	·	·	*	·	·	*	·	·	·	·	*	·	*	·	*			·	·	*	·	*	·	4
	4	6	5	9	8	7	13	10	3	26	21	15	11	22	27	23	2	20	16	25	14	1	18	17	19	12	24	

FIGURE C.

	20	21	8	11	3	4	15	19	23	25	17	18	22	24	16	14	7	6	1	12	13	10	9	2	5	27	26	
26	·	*	*	*	*	*	*	*	*	*	*	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	o	26
27	*	·	*	*	*	*	*	·	·	·	·	*	*	*	*	·	·	·	·	·	·	·	·	·	·	o	·	27
5	*	*	·	*	*	*	·	*	·	·	·	*	·	·	·	*	*	*	·	·	·	·	·	·	o	·	·	5
2	*	*	*	·	*	*	·	·	*	·	·	·	*	·	·	*	·	·	*	*	·	·	·	o	·	·	·	2
9	*	*	*	*	·	*	·	·	·	*	·	·	·	*	·	·	*	·	*	·	*	·	o	·	·	·	·	9
10	*	*	*	*	*	·	·	·	·	·	*	·	·	·	*	·	·	*	·	*	*	o	·	·	·	·	·	10
13	·	·	·	·	*	*	*	*	*	·	·	*	*	·	·	*	·	·	·	·	o	*	*	·	·	·	·	13
12	·	·	·	*	·	*	*	*	·	*	·	*	·	*	·	·	*	·	·	o	·	*	·	*	·	·	·	12
1	·	·	·	*	*	·	*	*	·	·	*	*	·	·	*	·	·	*	o	·	·	·	*	*	·	·	·	1
6	·	·	*	·	·	*	*	·	*	*	·	·	*	*	·	·	·	o	*	·	·	*	·	·	*	·	·	6
7	·	·	*	·	*	·	*	·	*	·	*	·	*	·	*	·	o	·	·	*	·	·	*	·	*	·	·	7
14	·	·	*	*	·	·	*	·	·	*	*	·	·	*	*	o	·	·	·	·	*	·	·	*	*	·	·	14
16	·	*	·	·	·	*	·	*	*	*	·	·	·	·	o	*	*	·	*	·	·	*	·	·	·	*	·	16
24	·	*	·	·	*	·	·	*	*	·	*	·	·	o	·	*	·	*	·	*	·	·	*	·	·	*	·	24
22	·	*	·	*	·	·	·	*	·	*	*	·	o	·	·	·	*	*	·	·	*	·	·	*	·	*	·	22
18	·	*	*	·	·	·	·	·	*	*	*	o	·	·	·	·	·	·	*	*	*	·	·	·	*	*	·	18
17	*	·	·	·	·	*	·	·	·	·	o	*	*	*	·	*	*	·	*	·	·	*	·	·	·	·	*	17
25	*	·	·	·	*	·	·	·	·	o	·	*	*	·	*	*	·	*	·	*	·	·	*	·	·	·	*	25
23	*	·	·	*	·	·	·	·	o	·	·	*	·	*	*	·	*	*	·	·	*	·	·	*	·	·	*	23
19	*	·	*	·	·	·	·	o	·	·	·	·	*	*	*	·	·	·	*	*	*	·	·	·	*	·	*	19
15	*	*	·	·	·	·	o	·	·	·	·	·	·	·	·	*	*	*	*	*	*	·	·	·	·	*	*	15
4	·	·	·	·	·	o	·	·	·	·	*	·	·	·	*	·	·	*	·	*	*	·	*	*	*	*	*	4
3	·	·	·	·	o	·	·	·	·	*	·	·	·	*	·	·	*	·	*	·	*	*	·	*	*	*	*	3
11	·	·	·	o	·	·	·	·	*	·	·	·	*	·	·	*	·	·	*	*	·	*	*	·	*	*	*	11
8	·	·	o	·	·	·	·	*	·	·	·	*	·	·	·	*	*	*	·	·	·	*	*	*	·	*	*	8
21	·	o	·	·	·	·	*	·	·	·	·	*	*	*	*	·	·	·	·	·	·	*	*	*	*	·	*	21
20	o	·	·	·	·	·	*	*	*	*	*	·	·	·	·	·	·	·	·	·	·	*	*	*	*	*	·	20
	20	21	8	11	3	4	15	19	23	25	17	18	22	24	16	14	7	6	1	12	13	10	9	2	5	27	26	

FIGURE D.

	6	22	7	15	13	18	8	2	3	10	11	9	4	27	20	5	21	16	23	26	17	1	19	12	25	14	24	
24	*	·	·	·	·	·	·	·	*	·	·	*	·	*	·	·	*	·	*	·	*	·	*	*	·	*	0	24
14	·	·	·	*	*	·	*	*	·	·	*	·	·	·	·	*	·	*	·	·	*	·	·	·	*	0	*	14
25	*	*	·	·	·	*	·	·	*	·	·	*	·	·	*	·	·	*	·	*	·	·	·	*	0	*	·	25
12	·	·	*	*	·	*	·	*	·	*	*	·	*	·	·	·	·	·	·	·	·	·	*	0	*	·	*	12
19	·	*	·	·	*	·	*	·	·	·	·	·	·	·	*	*	·	*	·	*	·	*	0	*	·	·	*	19
1	*	·	·	*	·	*	·	*	*	·	*	*	·	·	·	·	·	*	·	·	*	0	*	·	·	·	·	1
17	·	*	*	·	·	*	·	·	·	*	·	·	*	·	*	·	·	·	·	*	0	*	·	·	·	*	*	17
26	·	·	·	*	·	·	*	·	*	·	*	·	*	·	·	·	*	·	*	0	*	·	*	·	*	·	·	26
23	*	·	*	·	*	*	·	*	·	·	*	·	·	·	*	·	·	*	0	*	·	·	·	·	·	·	*	23
16	·	·	*	·	·	·	·	·	·	*	·	·	*	*	·	·	*	0	*	·	·	*	*	·	*	*	·	16
21	·	*	·	*	·	*	·	*	·	*	·	*	·	·	·	*	0	*	·	*	·	·	·	·	·	·	*	21
5	*	·	*	·	·	*	·	·	*	·	*	·	*	·	*	0	*	·	·	·	·	·	*	·	·	*	·	5
20	·	·	·	*	·	·	·	*	·	*	·	*	·	*	0	*	·	·	*	·	*	·	*	·	*	·	·	20
27	·	*	·	*	·	*	*	·	*	·	*	·	*	0	*	·	·	*	·	·	·	·	·	·	·	·	*	27
4	*	·	·	·	*	·	·	*	·	·	·	*	0	*	·	*	·	*	·	*	*	·	·	*	·	·	·	4
9	·	·	*	·	*	·	*	·	·	·	*	0	*	·	*	·	*	·	·	·	·	*	·	·	*	·	*	9
11	·	*	·	·	·	·	·	·	·	*	0	*	·	*	·	*	·	·	*	*	·	*	·	*	·	*	·	11
10	*	·	·	·	*	·	*	·	*	0	*	·	·	·	*	·	*	*	·	·	*	·	·	*	·	·	·	10
3	·	·	*	·	*	·	·	*	0	*	·	·	·	*	·	*	·	·	·	*	·	*	·	·	*	·	*	3
2	·	*	·	·	·	·	*	0	*	·	·	·	*	·	*	·	*	·	*	·	·	*	·	*	·	*	·	2
8	*	·	*	·	·	*	0	*	·	*	·	*	·	*	·	·	·	·	·	*	·	·	*	·	·	*	·	8
18	·	·	·	·	*	0	*	·	·	·	·	·	·	*	·	*	*	·	*	·	*	*	·	*	*	·	·	18
13	·	*	·	*	0	*	·	·	*	*	·	*	*	·	·	·	·	·	*	·	·	·	*	·	·	*	·	13
15	*	·	*	0	*	·	·	·	·	·	·	·	·	*	*	·	*	·	·	*	·	*	·	*	·	*	·	15
7	·	*	0	*	·	·	*	·	*	·	·	*	·	·	·	*	·	*	*	·	*	·	·	*	·	·	·	7
22	*	0	*	·	*	·	·	*	·	·	*	·	·	*	·	·	*	·	·	·	*	·	*	·	*	·	·	22
6	0	*	·	*	·	·	*	·	·	*	·	·	*	·	·	*	·	·	*	·	·	*	·	·	*	·	*	6
	6	22	7	15	13	18	8	2	3	10	11	9	4	27	20	5	2	16	23	26	17	1	19	12	25	14	24	

§ 1. If K, L, M, N, P, Q, R, S be eight linear functions of point coordinates in three dimensions, so that any one of them equated to zero represents a plane, then the equation

$$\mathrm{KLMN} = \theta \mathrm{PQRS} \quad \ldots \ldots \ldots \quad \text{(A)}$$

represents a quartic surface, which passes through each of the 16 straight lines given by the intersection of one plane from each of the groups, K, L, M, N and P, Q, R, S.

The equation contains $3 \times 8 + 1$, or 25 available constants.

Now if the planes be so related that the intersections of the pairs of planes K, P; L, Q; M, R; N, S, lie on a plane T, or, in other words, if the two tetrahedrons represented by the two sets of planes K, L, M, N and P, Q, R, S be in perspective, then, without further affecting the generality of the choice of the eight planes, we may assume

$$\mathrm{K} + \mathrm{P} \equiv \mathrm{L} + \mathrm{Q} \equiv \mathrm{M} + \mathrm{R} \equiv \mathrm{N} + \mathrm{S} \equiv \mathrm{T};$$

and the equation of the surface may be written

$$\mathrm{KLMN} = \theta (\mathrm{T} - \mathrm{K}) (\mathrm{T} - \mathrm{L}) (\mathrm{T} - \mathrm{M}) (\mathrm{T} - \mathrm{N}).$$

This is the equation of a quartic surface, which passes through 16 straight lines, and in which there are $3 \times 5 + 4 + 1$, or 20 available constants.

If, further, we take $\theta \equiv 1$, the term KLMN cancels, and the equation becomes divisible by T, the remaining factor equated to zero giving

$$\begin{aligned} &\mathrm{T}^3 - \mathrm{T}^2 (\mathrm{K} + \mathrm{L} + \mathrm{M} + \mathrm{N}) \\ &\quad + \mathrm{T} (\mathrm{KL} + \mathrm{KM} + \mathrm{KN} + \mathrm{LM} + \mathrm{MN} + \mathrm{NL}) \\ &\quad - (\mathrm{KLM} + \mathrm{LMN} + \mathrm{MNK} + \mathrm{NKL}) = 0 \quad \ldots \ldots \quad \text{(B)} \end{aligned}$$

the equation of a cubic surface, which passes through twelve straight lines,

.	L, P$^{(1)}$	M, P$^{(2)}$	N, P$^{(3)}$
K, Q$^{(6)}$	.	M, Q$^{(4)}$	N, Q$^{(5)}$
K, R$^{(8)}$	L, R$^{(9)}$	.	N, R$^{(7)}$
K, S$^{(10)}$	L, S$^{(11)}$	M, S$^{(12)}$	.

and which contains 19 available constants, the full number for the *general* equation of a cubic surface.

And since, if

$$\left.\begin{aligned} \mathrm{T} &= \mathrm{K} + \mathrm{L} \\ \mathrm{T} &= \mathrm{M} + \mathrm{N} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad \text{(C)},$$

then

$$T - K = L, \qquad T - L = K,$$
$$T - M = N, \qquad T - N = M,$$

it follows that the equations (C) satisfy equation (B) identically.

Now the equations (C) are equivalent to the equations

$$\left.\begin{array}{l} L - P \equiv K - Q = 0 \\ M - S \equiv N - R = 0 \end{array}\right\}.$$

Hence the straight line represented by these equations lies on the surface.

Similarly we see that the pairs of equations

$$\left.\begin{array}{l} T = K + M \\ T = L + N \end{array}\right\} \text{ and } \left.\begin{array}{l} T = K + N \\ T = L + M \end{array}\right\}$$

also satisfy equation (B) identically. Hence the straight lines, whose equations are

$$\left.\begin{array}{l} M - P \equiv K - R = 0 \\ N - Q \equiv L - S = 0 \end{array}\right\}$$

and

$$\left.\begin{array}{l} N - P \equiv K - S = 0 \\ M - Q \equiv L - R = 0 \end{array}\right\},$$

lie on the surface.

We have thus the equations of fifteen straight lines which lie on the cubic surface represented by equation

$$T^3 - T^2(K + L + M + N) + T(KL + KM + KN + LM + MN + NL)$$
$$- (KLM + LMN + MNK + NKL) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (B).$$

§ 2. Now, for convenience, let us take x, y, z, u instead of K, L, M, N, *i.e.*, let us choose the tetrahedron ABCD formed by the four planes K, L, M, N as the tetrahedron of reference.

Then we may represent the four planes P, Q, R, S by

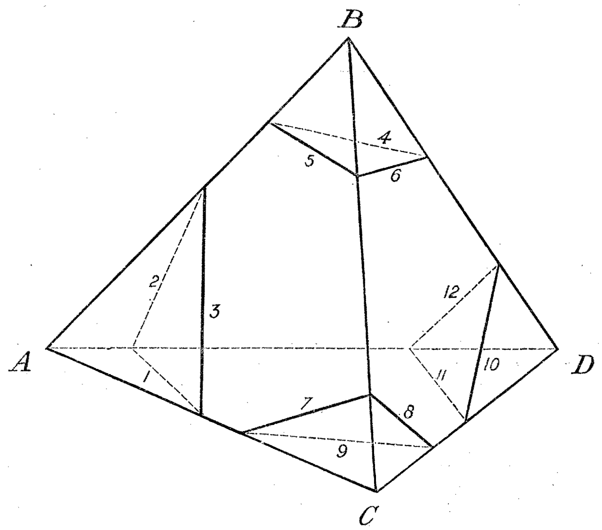
$$x = aT, \quad y = bT, \quad z = cT, \quad u = dT,$$

where $T \equiv \alpha x + \beta y + \gamma z + \delta u$, and where a, b, c, d, α, β, γ, δ are constants.

Then the equation (A) takes the form

$$xyzu = (x - a\mathrm{T})(y - b\mathrm{T})(z - c\mathrm{T})(u - d\mathrm{T}) \quad . \quad . \quad . \quad . \quad . \quad \text{(D)},$$

and it represents, besides the plane T, the cubic surface passing through the twelve straight lines, which are represented in the annexed figure, as well as three other straight lines which are not represented in the figure.



The equations of the lines may be written as follows :—

$$\left.\begin{matrix} x = a\mathrm{T} \\ y = 0 \end{matrix}\right\}(1) \qquad \left.\begin{matrix} x = a\mathrm{T} \\ z = 0 \end{matrix}\right\}(2) \qquad \left.\begin{matrix} x = a\mathrm{T} \\ u = 0 \end{matrix}\right\}(3)$$

$$\left.\begin{matrix} y = b\mathrm{T} \\ x = 0 \end{matrix}\right\}(6) \qquad \left.\begin{matrix} y = b\mathrm{T} \\ z = 0 \end{matrix}\right\}(4) \qquad \left.\begin{matrix} y = b\mathrm{T} \\ u = 0 \end{matrix}\right\}(5)$$

$$\left.\begin{matrix} z = c\mathrm{T} \\ x = 0 \end{matrix}\right\}(8) \qquad \left.\begin{matrix} z = c\mathrm{T} \\ y = 0 \end{matrix}\right\}(9) \qquad \left.\begin{matrix} z = c\mathrm{T} \\ u = 0 \end{matrix}\right\}(7)$$

$$\left.\begin{matrix} u = d\mathrm{T} \\ x = 0 \end{matrix}\right\}(10) \qquad \left.\begin{matrix} u = d\mathrm{T} \\ y = 0 \end{matrix}\right\}(11) \qquad \left.\begin{matrix} u = d\mathrm{T} \\ z = 0 \end{matrix}\right\}(12)$$

and

$$\left.\begin{matrix} \frac{x}{a} + \frac{u}{d} = \mathrm{T} \\ \frac{y}{b} + \frac{z}{c} = \mathrm{T} \end{matrix}\right\}(13),$$

which meets (3), (4), (9), and (10);

$$\left.\begin{array}{l}\frac{y}{b}+\frac{u}{d}-\mathrm{T} \\ \frac{x}{a}+\frac{z}{c}-\mathrm{T}\end{array}\right\} (14),$$

which meets (2), (5), (8), and (11); and

$$\left.\begin{array}{l}\frac{z}{c}+\frac{u}{d}-\mathrm{T} \\ \frac{x}{a}+\frac{y}{b}-\mathrm{T}\end{array}\right\} (15),$$

which meets (1), (6), (7), and (12).

§ 3. It is well known that every plane section of a cubic surface is a cubic curve. If, therefore, two straight lines be part of such a section, the remaining part of the section is a third straight line. If three straight lines form the section of a cubic surface by a plane, every other straight line on the surface must meet one of these lines and only one. We must, therefore, be able to construct all the remaining straight lines on the surface, by drawing all the straight lines which intersect each of the four triangles formed by the four sets of straight lines 1, 2, 3; 4, 5, 6; 7, 8, 9; and 10, 11, 12.

Now, since the twelve lines make triangles when taken also in the groups 6, 8, 10; 1, 9, 11; 2, 4, 12; and 3, 5, 7, it follows that every straight line on the surface must intersect one and only one from each of these groups.

Every remaining straight line on the surface must therefore intersect one line in each row, and one line in each column in the scheme

	1	2	3
6		4	5
8	9		7
10	11	12	

There are nine ways in which we can select one from each row and one from each column, viz. :—

1	4	7	10
1	5	8	12
1	6	7	12
2	5	8	11
2	5	9	10
2	6	7	11
3	4	8	11
3	4	9	10
3	6	9	12.

§ 4. In these groups there are distinct types of relation. Each of the three groups

1	6	7	12
2	5	8	11
3	4	9	10

represents two pairs of intersecting lines; for instance, the pair 3 and 10 intersect each other, and the pair 4 and 9 intersect each other, but neither 3 nor 10 intersects 4 or 9.

It is clear that the intersection of the plane containing the lines 3 and 10 and the plane containing the lines 4 and 9 meets the surface in *four* points, and therefore lies entirely on the surface.

Its equations are

$$\left.\begin{aligned}\frac{x}{a}+\frac{u}{d}&=\mathrm{T}\\ \frac{y}{b}+\frac{z}{c}&=\mathrm{T}\end{aligned}\right\}\quad(13).$$

In the same way it follows that the intersection of the planes of the lines 2, 8, and 5, 11, is a line on the surface, whose equations are

$$\left.\begin{aligned}\frac{y}{b}+\frac{u}{d}&=\mathrm{T}\\ \frac{x}{a}+\frac{z}{c}&=\mathrm{T}\end{aligned}\right\}\quad(14).$$

and that the intersection of the planes of the lines 1, 6, and 7, 12, is a line on the surface, whose equations are

$$\left.\begin{aligned}\frac{z}{c}+\frac{u}{d}&=\mathrm{T}\\ \frac{x}{a}+\frac{y}{b}&=\mathrm{T}\end{aligned}\right\}(15).$$

It will be observed that each of the lines 13, 14, and 15 lies in the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{u}{d}=2\mathrm{T}:$$

these three lines therefore meet each other and form a triangle.

§ 5. Each of the remaining six groups

1	4	7	10	(i.)
1	5	8	12	(ii.)
2	5	9	10	(iii.)
2	6	7	11	(iv.)
3	6	9	12	(v.)
3	4	8	11	(vi.)

represents a set of non-intersecting lines.

Two straight lines can be drawn to meet four non-intersecting straight lines; therefore two straight lines can be drawn to meet the lines of each group, and all straight lines so drawn will lie entirely on the surface. We are thus supplied with twelve more lines on the surface.

From what has preceded it will be clear that there is no other way of drawing a straight line on the surface. We have now obtained the whole of the twenty-seven lines which it is well known lie on the surface. The lines which meet the groups i., ii., iii., iv., v., vi., will be called 16, 17; 18, 19; 20, 21; 22, 23; 24 25; 26, 27 respectively.

§ 6. We will now proceed to find the equations of the lines 16, 17 which intersect the lines, 1, 4, 7 and 10.

Any line intersecting 1 and 7 is represented by equations of the form

$$\left.\begin{aligned}x-a\mathrm{T}&=\lambda y\\ z-c\mathrm{T}&=\mu u\end{aligned}\right\}(16)\text{ or }(17).$$

Since this line intersects (4) whose equations are

$$\left.\begin{aligned}y-b\mathrm{T}&=0\\ z&=0\end{aligned}\right\}$$

the equations

$$\begin{aligned} -x + (\lambda + a/b)\, y &= 0 \\ c/b\, y + \mu u &= 0 \\ \alpha x + (\beta - 1/b)\, y + \delta u &= 0 \end{aligned}$$

are simultaneously true.

Hence

$$\begin{vmatrix} -1 & \lambda + a/b & . \\ . & c/b & \mu \\ \alpha & \beta - 1/b & \delta \end{vmatrix} = 0.$$

Again, because this line intersects 10, whose equations are

$$\left.\begin{aligned} u - d\mathrm{T} &= 0 \\ x &= 0 \end{aligned}\right\},$$

the equations

$$\begin{aligned} \lambda y \quad + \quad a/d\,.\,u &= 0 \\ -\, z + (\mu + c/d)\, u &= 0 \\ \beta y + \gamma z + (\delta - 1/d)\, u &= 0 \end{aligned}$$

are satisfied simultaneously.

Hence

$$\begin{vmatrix} \lambda & . & a/d \\ . & -1 & \mu + c/d \\ \beta & \gamma & \delta - 1/d \end{vmatrix} = 0.$$

These equations of condition may be written as follows :—

$$\left.\begin{aligned} \alpha b\lambda\mu + (a\alpha + b\beta - 1)\,\mu - c\delta &= 0 \\ \gamma d\lambda\mu + (c\gamma + d\delta - 1)\,\lambda - a\beta &= 0 \end{aligned}\right\}.$$

It is clear that the values of λ and μ are the roots of the equations

$$\begin{aligned} c\gamma d\delta\lambda + (\alpha b\lambda + a\alpha + b\beta - 1)\,\{(c\gamma + d\delta - 1)\,\lambda - a\beta\} &= 0, \\ a\alpha b\beta\mu + (\gamma d\mu + c\gamma + d\delta - 1)\,\{(a\alpha + b\beta - 1)\,\mu - c\delta\} &= 0, \end{aligned}$$

respectively.

It is also clear that the roots of these equations must be so chosen that they satisfy the equation

$$(\alpha b\lambda\mu - c\delta)\,(\gamma d\lambda\mu - a\beta) = (a\alpha + b\beta - 1)\,(c\gamma + d\delta - 1)\,\lambda\mu,$$

which may be written

$$\alpha b\gamma d\lambda^2\mu^2 - (a\alpha b\beta + a\alpha c\gamma + a\alpha d\delta + b\beta c\gamma + b\beta d\delta + c\gamma d\delta$$
$$- a\alpha - b\beta - c\gamma - d\delta + 1)\,\lambda\mu + \alpha\beta c\delta = 0.$$

§ 7. Next we will find the equations of the lines 18 and 19, which meet the lines 1, 5, 8, 12.

Any line intersecting 1 and 12 is represented by equations of the form

$$\left.\begin{aligned} x - a\mathrm{T} &= \lambda y \\ u - d\mathrm{T} &= \nu z \end{aligned}\right\}(18) \text{ or } (19).$$

Since this line intersects (5), whose equations are

$$\left.\begin{aligned} y - b\mathrm{T} &= 0 \\ u &= 0 \end{aligned}\right\},$$

the equations

$$\begin{aligned} -x + (a/b + \lambda)\,y &= 0 \\ d/b\,y + \nu z &= 0 \\ \alpha x + (\beta - 1/b)\,y + \gamma z &= 0 \end{aligned}$$

are simultaneously true.

Hence

$$\begin{vmatrix} -1 & \lambda + a/b & . \\ . & d/b & \nu \\ \alpha & \beta - 1/b & \gamma \end{vmatrix} = 0.$$

Again, because this line intersects (8), whose equations are

$$\left.\begin{aligned} z - c\mathrm{T} &= 0 \\ x &= 0 \end{aligned}\right\},$$

the equations

$$\begin{aligned} \lambda y + a/c\,z &= 0 \\ (\nu + d/c)\,z - u &= 0 \\ \beta y + (\gamma - 1/c)\,z + \delta u &= 0 \end{aligned}$$

are simultaneously true.

Hence

$$\begin{vmatrix} \lambda & a/c & . \\ . & \nu + d/c & -1 \\ \beta & \gamma - 1/c & \delta \end{vmatrix} = 0.$$

These equations of condition may be written

$$\alpha b\lambda\nu + (a\alpha + b\beta - 1)\,\nu - \gamma d = 0$$
$$c\delta\lambda\nu + (c\gamma + d\delta - 1)\,\lambda - a\beta = 0$$

respectively.

Hence the values of λ and ν are the roots of the equations

$$c\gamma d\delta\lambda + (\alpha b\lambda + a\alpha + b\beta - 1)\,\{(c\gamma + d\delta - 1)\,\lambda - a\beta\} = 0$$
$$a\alpha b\beta\nu + (c\delta\nu + c\gamma + d\delta - 1)\,\{(a\alpha + b\beta - 1)\,\nu - \gamma d\} = 0$$

respectively.

It will be observed that the equation to find λ in determining the equations of 18 and 19 is identical with the equation to find λ in determining the equations of 16 and 17. It appears, therefore, that one of the two lines 18 and 19 lies in the plane of 1 and 16, and the other in the plane of 1 and 17. Here we assume that the complanar sets are 1, 16, 19, and 1, 17, 18.

In an exactly similar manner we can prove that each line of one pair intersects one or other of the lines of the second pair in the case of each of the sets of pairs—

i., iii.; i., iv.; i., vi.; ii., iii.; ii., v.; ii., vi.; iii., iv; iii., v.; iv., v.; iv., vi.; and v., vi.

§ 8. There are three other sets of pairs, to which a different method of proof must be applied, viz., i., v.; ii., iv. and iii., vi. Let us consider the lines of the pair v., that is, the lines 24 and 25, which intersect the lines 3, 6, 9 and 12.

Any line intersecting 3 and 9 is represented by equations of the form

$$\left.\begin{aligned} x - a\mathrm{T} &= \phi u \\ z - c\mathrm{T} &= \psi y \end{aligned}\right\}(24) \text{ or } (25).$$

Since this line intersects (6), whose equations are

$$\left.\begin{aligned} y - b\mathrm{T} &= 0 \\ x &= 0 \end{aligned}\right\},$$

the equations

$$a/b\,.\,y + \phi u = 0$$
$$(\psi + c/b)\,y - z = 0$$
$$(\beta - 1/b)\,y + \gamma z + \delta u = 0$$

are simultaneously true.

Hence

$$\begin{vmatrix} a/b & . & \phi \\ \psi + c/b & -1 & . \\ \beta - 1/b & \gamma & \delta \end{vmatrix} = 0.$$

Again, because this line intersects 12, whose equations are

$$\left.\begin{aligned} u - d\mathrm{T} &= 0 \\ z &= 0 \end{aligned}\right\},$$

the equations

$$-x + (\phi + a/d)\,u = 0,$$
$$\psi y + c/d\,u = 0,$$
$$\alpha x + \beta y + (\delta - 1/d)\,u = 0$$

are true simultaneously.

Hence

$$\begin{vmatrix} -1 & . & \phi + a/d \\ . & \psi & c/d \\ \alpha & \beta & \delta - 1/d \end{vmatrix} = 0.$$

These equations of condition may be written

$$b\gamma\phi\psi + (b\beta + c\gamma - 1)\,\phi - a\delta = 0,$$
$$\alpha d\phi\psi + (a\alpha + d\delta - 1)\,\psi - \beta c = 0,$$

respectively, and the values of ϕ, ψ must be so chosen that they satisfy the equation

$$(b\gamma\phi\psi - a\delta)\,(\alpha d\phi\psi - \beta c) = (a\alpha + d\delta - 1)\,(b\beta + c\gamma - 1)\,\phi\psi,$$

or

$$\alpha b\gamma d\phi^2\psi^2 - (a\alpha b\beta + a\alpha c\gamma + a\alpha d\delta + b\beta c\gamma + b\beta d\delta + c\gamma d\delta$$
$$- a\alpha - b\beta - c\gamma - d\delta + 1)\,\phi\psi + a\beta c\delta = 0.$$

It will be observed that the equation to find $\phi\psi$ in determining the equations of 24 and 25 is identical with the equation to find $\lambda\mu$ in determining the equations of 16 and 17.

Now, it is clear that the equations

$$\left.\begin{aligned} x - a\mathrm{T} &= \lambda y \\ z - c\mathrm{T} &= \mu u \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x - a\mathrm{T} &= \phi u \\ z - c\mathrm{T} &= \psi y \end{aligned}\right\}$$

are simultaneously true if $\lambda\mu = \phi\psi$. It follows, therefore, that each of the pair of lines 16, 17 cuts one or other of the pair 24, 25.

In an exactly similar manner we can prove that each line of one pair intersects one or other of the lines of the second pair in the case of each of the sets of pairs ii., iv., and iii., vi.

§ 9. We have thus shown that any one of the original twelve lines cuts ten others; the line 1, for instance, cuts 2, 3, 6, 9, 11, 15, 16, 17, 18, and 19.

Also we have shown that any one of the last twelve lines cuts nine others; 16, for instance, cuts 1, 4, 7, 10, and one from each of the pairs ii., iii., iv., v., vi. It must, therefore, cut one more, and that must be one from the group 13, 14, 15, since these three form a triangle.

The equations of 14 are

$$\left.\begin{aligned} \frac{y}{b} + \frac{u}{d} - \mathrm{T} = 0 \\ \frac{x}{a} + \frac{z}{c} - \mathrm{T} = 0 \end{aligned}\right\},$$

and the equations of 16 or 17 are

$$\left.\begin{aligned} x - a\mathrm{T} &= \lambda y \\ z - c\mathrm{T} &= \mu u \end{aligned}\right\}$$

where

$$\mathrm{T} = \alpha x + \beta y + \gamma z + \delta u,$$

and

$$\left.\begin{aligned} \alpha b\lambda\mu + (a\alpha + b\beta - 1)\,\mu - c\delta &= 0 \\ \gamma d\lambda\mu + (c\gamma + d\delta - 1)\,\lambda - a\beta &= 0 \end{aligned}\right\}.$$

If the lines intersect, the first five equations must be simultaneously true. Hence, eliminating x and z, we see that the equations

$$\frac{y}{b} + \frac{u}{d} - \mathrm{T} = 0$$

$$\frac{\lambda}{a}\,y + \frac{\mu}{c}\,u + \mathrm{T} = 0$$

$$(1 - a\alpha - c\gamma)\,\mathrm{T} = (\alpha\lambda + \beta)\,y + (\gamma\mu + \delta)\,u$$

are simultaneously true.

Hence

$$\begin{vmatrix} 1/b & 1/d & -1 \\ \lambda/a & \mu/c & 1 \\ \alpha\lambda + \beta & \gamma\mu + \delta & a\alpha + c\gamma - 1 \end{vmatrix} = 0,$$

or

$$\left(\frac{\alpha}{c} - \frac{\gamma}{a}\right)\lambda\mu + \frac{a\alpha + b\beta - 1}{bc}\,\mu - \frac{c\gamma + d\delta - 1}{ad}\,\lambda - \frac{\delta}{b} + \frac{\beta}{d} = 0,$$

which is identically true, as is at once seen by dividing the equations giving λ, μ by bc, ad respectively, and subtracting. This verifies the fact that each of the lines in the pair i. intersects 14.

Similarly it can be proved that each of the pair v. intersects 14, and that 13 intersects each of the lines in the pairs ii., iv., and 15 intersects each of the lines in the pairs iii. and vi.

We have now proved that of the twenty-seven lines on the cubic surface, each cuts ten of the others; furthermore we have shown which line cuts which others.

Now we might represent all the twenty-seven lines by their projections on a plane, where we should have to distinguish between the projection of the actual intersection of a pair of lines and the apparent intersection of the projections of two non-intersecting lines. We might from such a figure deduce many of the relations which exist between the lines; but the figure would be complicated, and the deductions would be attended with some difficulty.

§ 10. Now instead of this we will represent each line by one of a series of *parallel* straight lines in a plane, and we will then assume the figure turned round through a right angle, so that we have two lines representing each of the twenty-seven lines on the surface.

The intersection of two lines in the figure which represent the same line on the surface we mark with a zero.

The intersection of two lines, which represent two intersecting lines on the surface, we mark with a star, and the intersection of two lines, which represent two non-intersecting lines on the surface, is marked with a dot.

With this convention all the intersections of the twenty-seven lines on the surface are represented in Figure (A), in which each line is denoted by the number by which it has been known in the preceding investigation.

Of course it must be possible from such a figure to deduce all the relations which exist among the lines; but it will be found in actual practice that different transformations of the figure are more useful for different purposes.

§ 11. We will next point out the geometrical properties implied by certain combinations of the stars and dots which may occur in the figure.

Such a combination as

b	*	0
a	0	*
	a	b

implies that two lines intersect.

Here the rows and columns must represent the same lines.

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Such a combination as

c	*	*	0
b	*	0	*
a	0	*	*
	a	*b*	*c*

implies that three lines intersect each other in pairs, *i.e.*, that they form the complete section of the surface by their plane, which is a triple tangent plane.

Here, again, the rows and columns must represent the same lines.

Such a combination as

b	·	*
a	*	·
	c	*d*,

where the rows and the columns necessarily represent different lines, implies that a, c and b, d are intersecting pairs, and that b, c and a, d are non-intersecting pairs; but the figure does not indicate whether the pairs a, b and c, d intersect or do not intersect.

The whole truth with respect to the intersections of the four lines is not conveyed in the above figure.

When the whole truth is conveyed in the figure

d	·	*	·	0
c	*	·	0	·
b	·	0	·	*
a	0	·	*	·
	a	*b*	*c*	*d*,

that is, when there are no other intersections among the four lines than those represented in the figure

b	·	*
a	*	·
	c	*d*,

we shall call the combination a "double two."

Such a combination as

b	*	*
a	*	*
	c	*d*

where the rows and the columns necessarily represent different lines, implies that the lines *a*, *c*, *b*, *d*, taken in order, form a closed quadrilateral.

The whole truth with respect to these four lines is contained in this figure: no further truth is conveyed by the enlarged figure

d	*	*	·	0
c	*	*	0	·
b	·	0	*	*
a	0	·	*	*
	a	*b*	*c*	*d*.

Such a combination as

c	*	*	*
b	*	*	*
a	*	*	*
	d	*e*	*f*

implies that each of the three lines *a*, *b*, *c* intersects each of the three lines *d*, *e*, *f*.

It follows that the lines *a*, *b*, *c* are non-intersecting, and also that *d*, *e*, *f* are non-intersecting. This figure therefore conveys the whole truth with respect to the six lines.

We shall call such a set of six lines a "grille."

They form six of the generators of a hyperboloid of one sheet.

Such a combination as

c	·	*	*
b	*	·	*
a	*	*	·
	d	*e*	*f*

where the rows and the columns necessarily represent different lines, indicates that the six lines *a*, *d*, *b*, *f*, *c*, *e*, taken in order, form a closed hexagon.

We shall call such a set of six lines, if there are no other intersections, or if the whole truth with respect to their intersections is conveyed by the following figure, a "double three."

f	·	*	*	·	·	0
e	*	·	*	·	0	·
d	*	*	·	0	·	·
c	·	·	0	·	*	*
b	·	0	·	*	·	*
a	0	·	·	*	*	·
	a	*b*	*c*	*d*	*e*	*f*.

Such a combination as

d	·	*	*	*
c	*	·	*	*
b	*	*	·	*
a	*	*	*	·
	e	*f*	*g*	*h*

where the rows and the columns necessarily represent different lines, we shall call a "double four."

If any pair of non-intersecting lines, such as a, h be omitted, the remaining six form a closed hexagon, of which each of the omitted lines intersects three alternate sides.

The figure conveys the whole truth with respect to the intersections of the eight lines.

It may also be interpreted as representing a couple of closed quadrilaterals, a, e, b, f and c, g, d, h, each side of either of which intersects one—and only one—side of the other.

Such a combination as

e	·	*	*	*	*
d	*	·	*	*	*
c	*	*	·	*	*
b	*	*	*	·	*
a	*	*	*	*	·
	f	*g*	*h*	*i*	*j*

is called a "double five."

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Each line represented by a row or a column intersects four of the lines represented by the columns or the rows respectively.

The figure may be interpreted as representing a closed hexagon, say, *a*, *h*, *b*, *j*, *c*, *i*, and four lines, *d*, *e*, *f*, *g*, each of which intersects three alternate sides of the hexagon; or it may be interpreted as representing a "double four," together with two lines, say *e*, *f*, each of which cuts all the lines of one of the sets of four in the double four.

Such a combination as

f	·	*	*	*	*	*
e	*	·	*	*	*	*
d	*	*	·	*	*	*
c	*	*	*	·	*	*
b	*	*	*	*	·	*
a	*	*	*	*	*	·
	g	*h*	*i*	*j*	*k*	*l*

is called a "double six."

Each line represented by a row or a column intersects five of the lines represented by the columns or the rows respectively.

The figure may be interpreted as representing two "grilles," each line of either of which intersects two of the lines of the other; or, as representing two closed hexagons, each side of either of which intersects three alternate sides of the other.

The figure may be interpreted also as representing a "double four" and four lines, each of which intersects the four lines of one of the sets of the double four; or, again, as representing a "double five" and two lines, each of which intersects the five lines of one of the sets of the double five.

§ 12. From figure C we see that the number of lines which do not cut the line 26 is 16. Each of these sixteen lines has the same relation to the line 26; take any of them, say 27. Such a pair of lines as 26, 27 is called a "duad."

Again, from figure C, we see that the number of lines which do not cut the duad 26, 27 is 10. Each of these ten lines has the same relation to the duad; take any one of them, say 5. Such a set of lines as 26, 27, 5 is called a "triad."

Again, from figure C, we see that the number of lines which do not cut the triad 26, 27, 5 is 6. Each of these six lines has the same relation to the triad; take any one of them, say 2. Such a set of lines as 26, 27, 5, 2, is called a "tetrad."

Again, from figure C, we see that the number of lines which do not cut the tetrad 26, 27, 5, 2, is 3: the lines which do not cut are 9, 10, and 13. These three lines, however, have not all the same relation to the tetrad. The lines 9 and 10 have each one common line of intersection with the tetrad: in fact, the line 4 cuts the lines

26, 27, 5, 2, and 9, and the line 3 cuts the lines 26, 27, 5, 2, and 10; whereas both the lines 3 and 4 cut the lines 26, 27, 5, 2, and 13.

Such a set of lines as 26, 27, 5, 2, 9, is called a "pentad."

Again, from figure C, we see that there is but one line 10, which does not cut the pentad 26, 27, 5, 2, 9.

Such a set of lines as 26, 27, 5, 2, 9, 10, is called a "hexad."

We may summarize the last results by saying that the number of the lines of the surface which do not cut—

a single line on the surface		. . . .	is	16;
either line of a non-intersecting		duad . .	,,	10;
any	,,	,, triad . .	,,	6;
,,	,,	,, tetrad . .	,,	3;
,,	,,	,, pentad . .	,,	1;
,,	,,	,, hexad . .	,,	0.

Similarly, by inspection of the top six rows of Figure C, we conclude that—

10 lines on the surface cut a definite line on the surface, and 16 do not.

5 lines cut both the lines of a duad; 10 lines cut 1; and 10 cut neither.

3 lines cut all the lines of a triad; 6 lines cut 2; 9 lines cut 1; and 6 cut none.

2 lines cut all the lines of a tetrad; 4 lines cut 3; 6 lines cut 2; 8 lines cut 1; and 3 cut none.

1 line cuts all the lines of a pentad; 5 lines cut 4; 10 cut 2; 5 cut 1; and 1 cuts none.

No lines cut all the lines of a hexad; 6 lines cut 5; 15 cut 2; none cut 1 only; and none cut none.

We are now enabled to find the number of different duads, triads, &c.

$$\text{Number of duads} \quad . \quad . = \frac{27 \cdot 16}{1 \cdot 2} = 216;$$

$$\text{,, \quad triads} \quad . \quad . = \frac{27 \cdot 16 \cdot 10}{1 \cdot 2 \cdot 3} = 720;$$

$$\text{,, \quad tetrads} \quad . \quad . = \frac{27 \cdot 16 \cdot 10 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 1080;$$

$$\text{,, \quad pentads} \quad . \quad . = \frac{27 \cdot 16 \cdot 10 \cdot 6 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 432;$$

$$\text{,, \quad hexads} \quad . \quad . = \frac{27 \cdot 16 \cdot 10 \cdot 6 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 72.$$

The results of § 12 are all to be found in STURM, 'Synthetische Untersuchungen über Flächen Dritter Ordnung.'

§ 13. It is well known that the number of triangles on a cubic surface is 45.

We may calculate the number of closed quadrilaterals, pentagons, and hexagons, restricting the denomination to polygons of the proper number of sides, no two sides of which intersect each other except consecutive sides.

By inspection of one of the figures (and for this purpose Figure D is the most convenient) it is easy to see that the number of lines on the surface which intersect

both lines of an open angle is	1	(see lines 14, 24),
both the end lines and no others of an open trilateral is . .	4	(see lines 25, 14, 24),
,, ,, ,, ,, quadrilateral is .	3	(see lines 19, 12, 25, 14),
,, ,, ,, ,, quinquilateral is	1	(see lines 17, 1, 19, 12, 25),

and that the number of lines on the surface which intersect only one line, and that,

a specified end line of an open angle, is	8
only one line (an end line) of an open trilateral, is . . .	4
,, ,, ,, quadrilateral, is . .	1
,, ,, ,, quinquilateral, is .	0,

and that the number of lines on the surface which intersect none of the lines

of an open angle, is	8
,, trilateral, is	4
,, quadrilateral, is	3
,, quinquilateral, is . .	3.

(An open sexilateral does not exist on the surface.)

By means of Figure D we can see, by inspection of the lines 8, 2, 3, 10, that they form a closed quadrilateral, and that some one of them is intersected by every other line except 15.

By inspection of the lines 13, 18, 8, 2, 3, that they form a closed pentagon, and that some one of them is intersected by every other line except 11 and 16, which do not intersect.

By inspection of the lines 2, 3, 10, 11, 9, 4, that they form a closed hexagon, and that some one of them is intersected by every other line except 15, 18, 19, which do not intersect.

It appears, therefore, that there is but one line on the surface which does not intersect one line at least of a closed quadrilateral on the surface; that there are two lines only, forming a non-intersecting duad, which do not intersect one line at least of a closed pentagon on the surface; and that there are three lines only, forming a non-intersecting triad, which do not intersect one line at least of a closed hexagon on the surface.

§ 14. *Closed Quadrilaterals.*—If the lines a, b, c, d, taken in order form a closed quadrilateral, it appears from what has gone before that

when a is given, there are 10 ways of choosing b;

when a, b are given, there are 8 ways of choosing c; and that

when a, b, c are given, there are 4 ways of choosing d.

Hence, the number of orders of choosing 4 lines to form a quadrilateral is 27 . 10 . 8 . 4, and each quadrilateral will appear 4×2 or 8 times.

The total number of closed quadrilaterals therefore is

$$\frac{27 . 10 . 8 . 4}{4 . 2} = 1080.$$

Now we have shown that there is only one line which does not cut at least one of the sides of a closed quadrilateral.

There must, therefore, be $1080/27 = 40$ closed quadrilaterals which each line does not cut.

There are 16 lines which do not cut a given line; therefore these 40 quadrilaterals are formed of 16 lines, and these 16 lines are capable of being divided into sets of four quadrilaterals in ten different ways.

One such set of four quadrilaterals, none of the sides of which cut the given line 26, is 27, 21, 5, 18; 2, 22, 13, 14; 9, 24, 12, 7; 10, 16, 1, 6.

§ 15. *Closed Pentagons.*—If the lines a, b, c, d, e, taken in order form a closed pentagon, it appears that

when a is given, the number of ways of choosing b is 10;

when a and b are given, the number of ways of choosing c is 8;

when a, b, and c are given, the number of ways of choosing d is 4;

when a, b, c, and d are given, the number of ways of choosing e is 3.

Hence the number of orders of choosing five lines to form a closed pentagon is 27 . 10 . 8 . 4 . 3, and each pentagon will appear 5×2 or 10 times.

The total number of closed pentagons therefore is $27 . 8 . 4 . 3 = 2592$.

Now we have shown that there are only two lines, forming a non-intersecting duad, which do not cut one at least of the sides of a closed pentagon.

There must, therefore, be 2592/216 = 12 closed pentagons for each duad.

There are ten lines which do not cut either of the lines of a duad.

Therefore these twelve pentagons are formed of ten lines, and these ten lines form pairs of pentagons in six different ways.

One such pair of pentagons, none of the sides of which cut either of the lines 26, 27, is

18, 5, 14, 2, 12 and 24, 9, 13, 10, 6.

§ 16. *Closed Hexagons.*—If the lines *a*, *b*, *c*, *d*, *e*, *f*, taken in order form a closed hexagon, it appears that when *a* is given the number of ways of choosing *b*, *c* and *d* is 10 . 8 . 4, when *a*, *b*, *c*, *d* are given, the number of ways of choosing *e* is 1, and that when *a*, *b*, *c*, *d* and *e* are given, the number of ways of choosing *f* is 1.

Hence the number of orders of choosing six lines to form a closed hexagon is

$$27 \cdot 10 \cdot 8 \cdot 4 \cdot 1 \cdot 1,$$

and each hexagon will appear $6 \times 2 = 12$ times.

The total number of closed hexagons, therefore, is 9 . 10 . 8 = 720.

Now we have shown that there are only three lines, forming a non-intersecting triad, which do not cut one at least of the sides of a closed hexagon.

There must, therefore be 720/720 = 1 closed hexagon for each triad.

There are six lines which do not cut any of the lines of a triad.

Therefore, there is but one closed hexagon formed of the six lines which do not cut a triad.

The hexagon, none of whose sides cut any of the lines 26, 27, 5 is 1, 2, 12, 10, 13, 9.

If *a*, *b*, *c*, *d*, *e*, *f* be the sides of a closed hexagon in order, every line on the surface which does not meet *a*, *b*, *c*, *d*, *e* or *f*, must meet the lines which meet the pairs *a*, *b* ; *b*, *c* ; *c*, *d* ; *d*, *e* ; *e*, *f* ; *f*, *a*.

Now the intersection of the planes *a*, *b* and *d*, *e*, is a line on the surface ; that is, the lines joining *a*, *b* ; *b*, *c* ; *c*, *d* are identical with those joining *d*, *e* ; *e*, *f* ; *f*, *a* respectively ; and the three form a non-intersecting triad.

Three other lines, forming a non-intersecting triad, meet them, and they are the three lines each of which misses each side of the closed hexagon.

§ 17. From the closed hexagon, formed of the lines *a*, *b*, *c*, *d*, *e*, *f*, we can form six planes, *ab*, *bc*, *cd*, *de*, *ef*, *fa*, such that the planes *ab*, *cd*, *ef* intersect the planes *bc*, *de*, *fa*, in nine of the twenty-seven lines.

Hence the number of ways of throwing the equation of a cubic surface into the form LMN = PQR, may be found as follows :

From each such form of the equation we can obtain six closed hexagons, and from each closed hexagon we can obtain one such form of equation.

Hence, the number of such forms of equation

$$= \tfrac{1}{6} \times \text{the number of closed hexagons}$$
$$= \tfrac{1}{6} \,.\, 720 = 120.^{*}$$

§ 18. In the case of a double two,

the planes a, c, and b, d, are both triple tangent planes.

b	·	*
a	*	·
	c	d

The intersection of these planes has clearly four points on the surface; it is, therefore one of the twenty-seven lines.

Hence, for each line on the surface there are $5.4/1.2 = 10$ pairs of triangles, each of which gives a double two. But if we reckon the two figures

b	·	*
a	*	·
	c	d

d	·	*
a	*	·
	c	b

which represent the same set of four lines if they are double twos, as distinct double twos; we say the number of double twos

$$= 27 \,.\, 10 \,.\, 2 = 540.$$

From a double three we can obtain three double twos; this is seen at once, for in a double three, such as

c	·	*	*
b	*	·	*
a	*	*	·
	d	e	f

we can leave out either of the pairs a, f; b, e; or c, d; and from Figure C, we see at once that we can from a double two form four double threes.

Hence the number of double threes

$$= 4/3 \times \text{the number of double twos}$$
$$= 4/3 \,.\, 540 = 4 \,.\, 180 = 720.$$

* This number is given in SALMON, 'Solid Geometry,' 3rd edition, p. 466.

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Similarly we see that from a double four

we can form four double threes

d	·	*	*	*
c	*	·	*	*
b	*	*	·	*
a	*	*	*	·
	e	f	g	h

and from Figure C we see that from a double three we can form three double fours.

Hence the number of double fours

$$= \tfrac{3}{4} \times \text{the number of double threes}$$
$$= \tfrac{3}{4} \,.\, 720 = 3 \,.\, 180 = 540.$$

Similarly from each double five we can form five double fours, and from each double four we can form two double fives.

Hence the number of double fives

$$= \tfrac{2}{5} \times \text{the number of double fours}$$
$$= \tfrac{2}{5} \,.\, 540 = 2 \,.\, 108 = 216.$$

Similarly from each double six we can form six double fives, and from each double five we can form one double six.

Hence the number of double sixes

$$= \tfrac{1}{6} \times \text{the number of double fives}$$
$$= \tfrac{1}{6} \,.\, 216 = 36.^{*}$$

§ 19. Now let us choose one triple tangent plane, say the plane through the lines

$$4 \quad , \quad 6 \quad , \quad 5 \,;$$

twelve other triple tangent planes pass through one or other of these lines.

The remaining 45–13 or 32 planes all hold a similar relation to the first plane.

Let us choose, as a second plane, one of those thirty-two planes, say the plane through the lines

$$9 \quad , \quad 8 \quad , \quad 7.$$

With respect to the two triple tangent planes which do not pass through a line in

* This result was obtained first by SCHLÄFLI.

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common, there are twenty-two triple tangent planes which have no line in common with the first two planes.

This result may be obtained by counting the triple tangent planes which do not contain any of the six lines 4, 5, 6, 7, 8 or 9, or it may be calculated otherwise.

§ 20. But among these twenty-two planes, there are three distinct types of relationship to the first pair of planes.

The only type with which we are here concerned, is that in which the first line of the third plane cuts the first line of the second and of the third planes; the second line cuts the second lines, and consequently the third line cuts the third lines.

In this case, the first lines form a triple tangent plane, as do also the second lines and the third lines.

In Figure B it is easily seen that the nine lines

4 ,	6 ,	5
9 ,	8 ,	7
13 ,	10 ,	3

give triple tangent planes when the numbers are read either horizontally or vertically.

The only B triangles which do not contain any of the lines 4, 6, 5, 9, 8, 7, 13, 10, 3, are as follows :—

1 ,	16 ,	19
1 ,	17 ,	18
2 ,	21 ,	22
2 ,	20 ,	23
11 ,	22 ,	27
11 ,	23 ,	26
12 ,	18 ,	25
12 ,	19 ,	24
14 ,	16 ,	25
14 ,	17 ,	24
15 ,	21 ,	26
15 ,	20 ,	27.

If towards completing a set of triangles we select the triangle

1 , 16 , 19,

we must take also

12 , 18 , 25

and

14 , 17 , 24,

and similarly, if we select the triangle

$$2\ ,\ 21\ ,\ 22,$$

we must take also

$$11\ ,\ 23\ ,\ 26$$

and

$$15\ ,\ 20\ ,\ 27.$$

We can see that if we were to choose the triangle

$$1\ ,\ 17\ ,\ 18,$$

we must also take

$$12\ ,\ 19\ ,\ 24,$$

and

$$14\ ,\ 16\ ,\ 25\ ;$$

and if we select the triangle

$$2\ ,\ 20\ ,\ 23,$$

we must take also

$$11\ ,\ 22\ ,\ 27,$$

and

$$15\ ,\ 21\ ,\ 26.$$

Hence we see that the three groups

$$\begin{vmatrix} 4 & 6 & 5 \\ 9 & 8 & 7 \\ 13 & 10 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 16 & 19 \\ 17 & 14 & 24 \\ 18 & 25 & 12 \end{vmatrix} \cdot \begin{vmatrix} 2 & 21 & 22 \\ 20 & 15 & 27 \\ 23 & 26 & 11 \end{vmatrix} .$$

form three sets such that the triangle obtained by reading any row or column is of the type we have considered above, with respect to the triangles obtained by reading the other two rows or columns, and also that there is but one way of completing the second and third sets when the first is chosen.

§ 21. Two triple tangent planes, which do not pass through the same line, intersect in a straight line which cuts the two triangles in the same three points, these points being intersections of pairs of lines on the surface.

There are $45.16 = 720$ such pairs of triple tangent planes; there are, therefore, 720 straight lines which run through three of the points of contact of the triple tangent planes, and no more.

Each set of three points in a small square in Figure B gives three points, which are the intersections of the sides of two triangles. Each set, therefore, lies on a straight line.

§ 22. Each pair of triangles which do not have a common line, such as 1, 2, 3 and 6, 8, 10, gives three complete schemes for a pair of tetrahedrons in perspective, viz.:

.	1	2	3	.	1	2	3	and	.	1	2	3
6	.	4	5	6	.	23	24		6	.	22	25
8	9	.	7	8	18	.	27		8	19	.	26
10	11	12	.	10	17	20	.		10	16	21	.

and each pair of tetrahedrons gives four pairs of such triangles.

Therefore the number of pairs of tetrahedrons

$$= \tfrac{3}{4} \times \text{the number of pairs of such triangles}$$
$$= \tfrac{3}{4} \,.\, 45 \,.\, 32/2 = 3 \,.\, 45 \,.\, 4 = 6 \,.\, 90 = 540.$$

It follows that the line, which is the intersection of the planes 1, 2, 3 and 6, 8, 10, lies in a plane with the intersections of the pairs of planes

	4,	5,	6	and	1,	9,	11
	7,	8,	9	and	2,	4,	12
and	10,	11,	12	and	3,	5,	7.

There are, therefore, three distinct planes of perspective passing through each of the 720 lines, and each perspective plane passes through four of the lines.

§ 23. From a closed quadrilateral, such as

2	3
4	5

we can, by choosing the four lines which cut two consecutive sides of the quadrilateral, obtain the figure

.	.	7	.
1	2	3	.
.	4	5	6
.	12	.	.

This we can complete in three distinct ways by filling up the corner spaces, so that the rows and the columns will all give triple tangent planes.

The figures are as follows :—

$$\begin{vmatrix} 9 & . & 7 & 8 \\ 1 & 2 & 3 & . \\ . & 4 & 5 & 6 \\ 11 & 12 & . & 10 \end{vmatrix}$$

$$\begin{vmatrix} 16 & . & 7 & 23 \\ 1 & 2 & 3 & . \\ . & 4 & 5 & 6 \\ 19 & 12 & . & 24 \end{vmatrix}$$

and

$$\begin{vmatrix} 17 & . & 7 & 22 \\ 1 & 2 & 3 & . \\ . & 4 & 5 & 6 \\ 18 & 12 & . & 25 \end{vmatrix}.$$

This proves that from every closed quadrilateral we can obtain three distinct pairs of tetrahedrons in perspective, and, therefore, three distinct perspective planes.

If we have two triple tangent planes which do not possess a line in common, say,

$$1 \quad 2 \quad 3$$

and

$$6 \quad 4 \quad 5$$

we can obtain from them in nine different ways a pair of tetrahedrons in perspective, and from every pair of such tetrahedrons we can obtain $4.3 = 12$ pairs of such triple tangent planes.

Therefore, the number of perspective planes

$$= \text{the number of pairs of tetrahedrons in perspective,}$$

$$= \tfrac{9}{12} \times \text{the number of such pairs of triple tangent planes,}$$

$$= \tfrac{3}{4} \cdot \tfrac{45 \cdot 32}{2} = 3 \;.\; 45 \;.\; 4 = 6 \;.\; 90 = 540.$$

A set of lines such that they form triangles when read in rows or columns, as

$$\begin{matrix} a, & b, & c \\ d, & e, & f \\ g, & h, & i \end{matrix}$$

is obtainable from every one of the possible forms of the equation of the cubic surface, such as LMN = PQR.

There are 120 such sets (§ 17), and when one is chosen there is only one way of completing the set of triangles by similar sets of nine lines (§ 20). Therefore, there must be 40 different ways in which all the lines on the surface can be arranged, such as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} j & k & l \\ m & n & o \\ p & q & r \end{vmatrix} \qquad \begin{vmatrix} s & t & u \\ v & w & x \\ y & z & \omega \end{vmatrix}$$

such that each row and each column of any one of the three sets gives a triangle.

The number may also be calculated by considering how many such sets as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

exist containing a definite line a.

There are five triangles which contain a. There are, therefore, ten pairs of such triangles, or ten selections of a, b, c, d, g in the set; for each pair of b and d there are four lines which could take the place of e, and then the set is determined uniquely.

There are, therefore, $10 \times 4 = 40$* such sets.

* Sturm, 'Synth. Unters. über Flächen Dritter Ordnung.'